Real-time estimation of kinematic variables for advanced mechanical systems

[Workshop] Dynamics and control of classical and quantum systems

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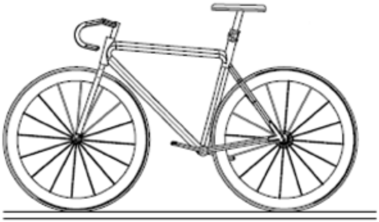
Mechanical systems

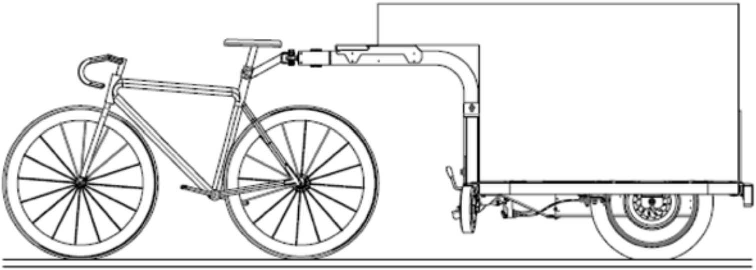
- Essential quantities of mechanical systems include generalized coordinates (positions, velocities, angles, angular velocities, *motional variables* for non-holonomic constraints), inertia, as well as internal and external forces
- Mastering the motion to carefully selected reference signals
- Typical instruments include accelerometers, gyroscopes, strain gauges, radar, magnetometers, resonators, Pitot tubes, PIV (Particle Image Velocimetry), ...
- Core claim: stabilize the unstable, accelerate the sluggish, unburden the heavy

Dynamics transformation by feedback on internal forces



Operates on a simple open-loop principle: a sensor in crankset detects rotation, triggers the hub motor, monitors rotation speed and contributes to propulsion based on look-up tables ("Eco" / "Comfort" / "Turbo")

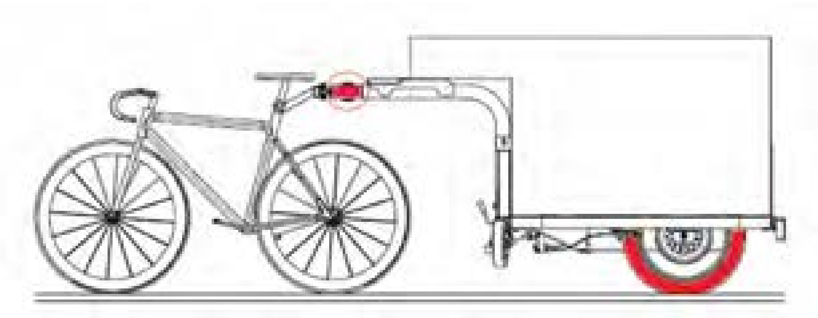
Dynamics transformation by feedback on internal forces



Attach a heavy trailer (100-500 Kg) and try to make it transparent

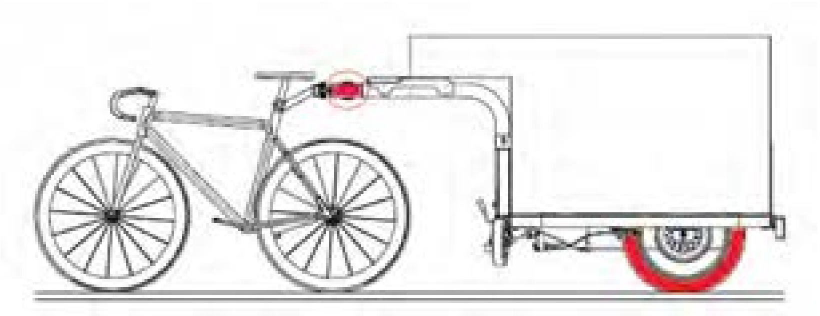
The basic strategy is not sufficient. “Turbo” on all the time.

Dynamics transformation by feedback on internal forces



Trailer has powerful motors, mechanical interaction occurs only at the drawbar

Dynamics transformation by feedback on internal forces

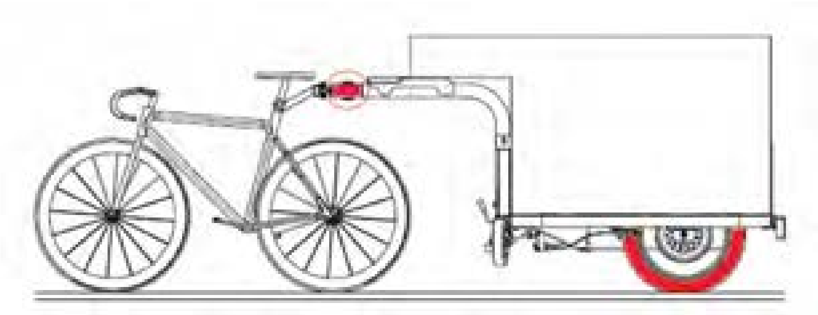


$$\dot{x} = f(x, \underbrace{u_1}_{\text{cyclist}}, \underbrace{u_2}_{\text{electric drives}}, \underbrace{p}_{\text{(masses, geometry, ...)}}), \quad y = h(x, u, p)$$

Model assignment - “dynamics shaping”

$$\begin{cases} \dot{x} = f(x, u_1, k(\xi), p) \approx f(x, u_1, 0, p_0) : \text{bike only} \\ \dot{\xi} = g(\xi, y) \end{cases}$$

Dynamics transformation by feedback on internal forces



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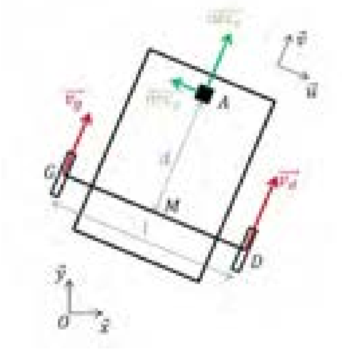
Mass cancellation is achieved

Estimating intern. forces, extern. forces, and user intention

longit. strain gauge in the drawbar



velocities, accelerometers

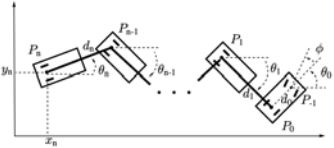
Generalization of dynamic assignment via feedback

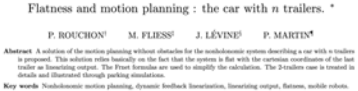
intent amplification, kinematic equivalences (geometry) in cornering.

A commercial product: context detection (slope, slippery roads, falls, sensor failure, battery ageing,...)

K-Ryole in Paris (10 years of R&D, 1500 vehic.)

The *n* trailers





Flatness and motion planning : the car with n trailers. *

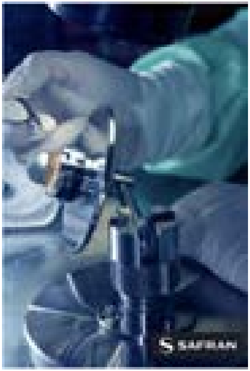
P. ROUCHON[1] M. FLIESS[2] J. LÉVINE[3] P. MARTIN[4]

Abstract A solution of the motion planning without obstacles for the nonholonomic system describing a car with n trailers is proposed. This solution relies basically on the fact that the system is flat with the cartesian coordinates of the last trailer as linearizing output. The Frnet formulas are used to simplify the calculation. The 2-trailers case is treated in details and illustrated through parking simulations.

Key words Nonholonomic motion planning, dynamic feedback linearization, linearizing output, flatness, mobile robots.

1992 paper, a lasting inspiration

Inertial measurement units (IMUs), mutliple axis



Safran HRG

SBG

Bosch BNO055

Weaknesses: vibrational/acoustic disturbances, converter flips ...

Prices: $\times\frac{1}{50}$ from left to right. Availability

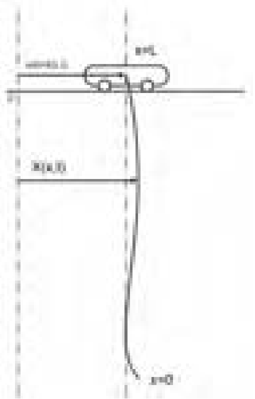
MEMS (microfab. resonators)

What can you achieve with these? (by combining them?)

Problem 1: propagation through a Kinematic Chain

Idea: deploy IMUs massively

Equip a mechanical system with distributed inertial sensors



heavy chain $\frac{\partial}{\partial x}\left(gx\frac{\partial X}{\partial x}\right) - \frac{\partial^2 X}{\partial t^2} = 0$

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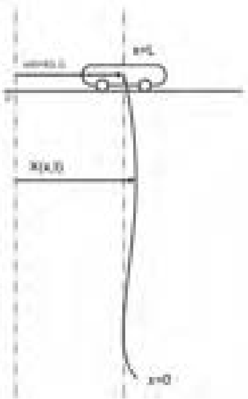

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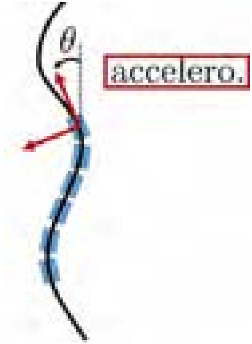
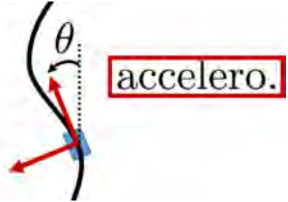


sensors

Accelerometer measurements

$$M = \begin{pmatrix} X(x,t) \\ x \end{pmatrix}, \quad \tan\theta = -\frac{\partial X}{\partial x}, \quad \frac{\partial}{\partial x}\left(gx\frac{\partial X}{\partial x}\right) - \frac{\partial^2 X}{\partial t^2} = 0$$

$$\begin{aligned}
\text{accelero.} &= R^T_{\theta+\pi/2}\left(\ddot{M} + \begin{pmatrix} 0 \\ g \end{pmatrix}\right) \\
&\approx \begin{pmatrix} g \\ -\frac{\partial^2 X}{\partial t^2} + g\frac{\partial X}{\partial x} \end{pmatrix} \quad \text{(small angles)} \\
&= \begin{pmatrix} g \\ -gx\frac{\partial^2 X}{\partial x^2} \end{pmatrix} = \ldots \begin{pmatrix} g \\ -\frac{g}{\tau'(x)}\tau(x)\frac{\partial^2 X}{\partial x^2} \end{pmatrix} \\
\text{gyro.} &= \dot{\theta} = -\frac{\partial^2 X}{\partial x \partial t} \\
\text{strain g.} &= -\frac{\partial^2 X}{\partial x^2}
\end{aligned}$$



State reconstruction from accelerometer measurements

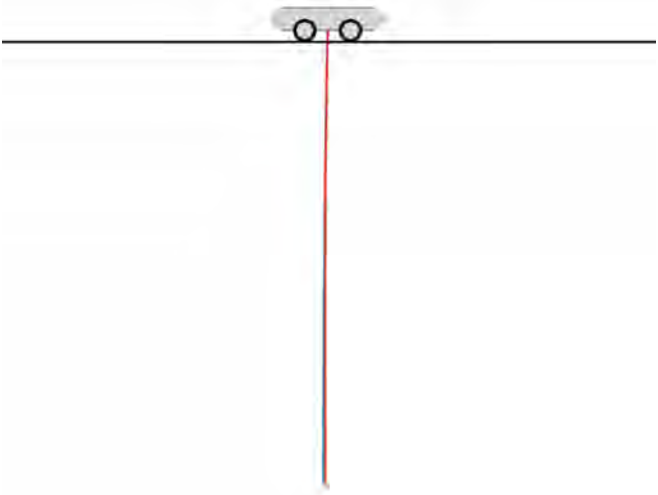
$$\text{accelero.}(x,t) = -gx\frac{\partial^2 X}{\partial x^2}(x,t)$$

Summation of distributed accelero.

$$\frac{\partial X}{\partial x}(x,t) = \frac{\partial X}{\partial x}(L,t) + \int_x^L \frac{\text{accelero.}(\ell,t)}{g\ell}d\ell$$

$$X(x,t) = X(L,t) - \int_x^L \frac{\partial X}{\partial x}(\ell,t)d\ell$$

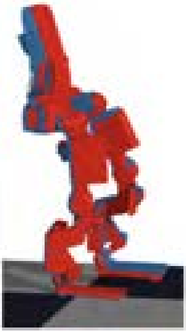
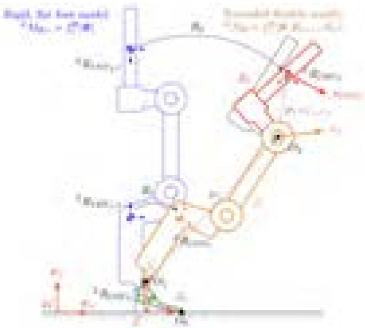
Simulation



The real case study: Wandercraft exoskeletton

- Gyroscope
- ~~Magnetometer~~
- Accelerometer

NS, LP, FdM, NP, +

Flexibilities

Reference frames

Positions or rigid bodies (local reference frame)

Kinematic chain from stance foot to swing foot

Not a standard robot: carries a non instrumented patient

System flexes, foot deviates, ground contact occurs too early, wrong mass transfer, ...

Observer

For any point M in the kinematic chain

$$M = f_{flex.}(\text{encoders}, R_0, R_1, ..., R_n)$$

Measurement equation (not only gravity)

$$\begin{cases} \mathbf{y}_{a,i} = R_{IMU,i}^T \left(\ddot{\mathbf{p}}_{IMU,i} - \mathbf{g} \right) \\ \mathbf{y}_{g,i} = \mathbf{w}_{IMU,i}, \qquad \dot{R}_{IMU,i} = R_{IMU,i} \left[\mathbf{w}_{IMU} \right]_\times \end{cases}$$

Velocity of each IMU (in its own reference frame)

$$\mathbf{v}_{IMU,i} = R_{IMU,i}^T \, \dot{\mathbf{p}}_{IMU,i}$$

$$\dot{\mathbf{v}}_{IMU,i} = -[\mathbf{y}_{g,i}]_\times \, \mathbf{v}_{IMU,i} + \mathbf{y}_{a,i} + R_{IMU,i}^T \, \mathbf{g}$$

Approximation through a chain rule (from the stand foot)

$$\mathbf{v}^{\text{approx.}} : \quad \begin{cases} \mathbf{v}_{O_i} = \mathcal{P}_{i-1} \mathbf{v}_{O_{i-1}} + \text{compo.}(\mathbf{y}_g, \text{encoders}) \times \ell_i \\ \mathbf{v}_{IMU,i} = \mathcal{Q}_i \mathbf{v}_{O_i} + [\mathbf{y}_g]_\times \, \ell_{IMU_i/O_i} \end{cases}$$

Observer (cont.)

Tilt vector (only relevant information, reduction of unknowns)

$$\mathbf{t} = R^T \mathbf{e_z}$$

For each rigid body in the kinematic chain

$$\begin{cases} \dot{\mathbf{v}} = -[\mathbf{y}_g]_\times \mathbf{v} + \mathbf{y_a} + g\,\mathbf{t} \\ \dot{\mathbf{t}} = -[\mathbf{y}_g]_\times \mathbf{t} \end{cases}$$

Cascade calculus from the ground to the swing foot

$$\begin{cases} \dot{\hat{\mathbf{v}}} = -[\mathbf{y}_g]_\times \hat{\mathbf{v}} + \mathbf{y_a} + g\,\hat{\mathbf{t}} + \alpha(\hat{\mathbf{v}} - \mathbf{v}^{\mathrm{approx.}}) \\ \dot{\hat{\mathbf{t}}} = -[\mathbf{y}_g - \beta[\hat{\mathbf{t}}]_\times (\hat{\mathbf{v}} - \mathbf{v}^{\mathrm{approx.}})]_\times \hat{\mathbf{t}} \end{cases}$$

$\alpha > \sqrt{\beta g}$, practical convergence (exponential, residual error, high gain)

$$\hat{R} = \exp\left[\arccos(\mathbf{e_z}^T \hat{\mathbf{t}}) \frac{\mathbf{e_z} \times \hat{\mathbf{t}}}{\left\|\mathbf{e_z} \times \hat{\mathbf{t}}\right\|}\right]_\times R_{\mathrm{rigid}}$$

Estimate and compensate flexibility-induced foot deviations

Robotics everywhere?

The missing ingredient: force control (not only position)

“Stick-slip” is prevalent, 3D, varying

Advanced metrology

Not only alter system behavior but identify its parameters

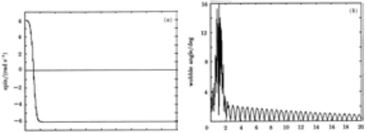
An example of advanced metrology: Ballistic wind tunnels

Before, let us discuss familiar complex mechanical systems

Celtic stone (rattleback)

(handcrafted by Pierre)

Reverses direction of rotation

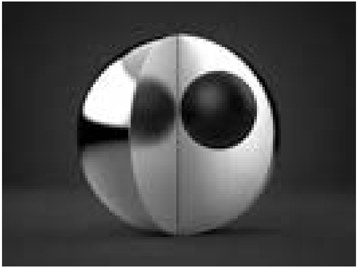


[1] Garcia, Hubbard "Spin Reversal of the Rattleback: Theory and Experiment". Proc. of the Royal Society A (1988)

Routh sphere

Dynamically non-symmetric, unbalanced (displaced center of mass) ball rolling without slipping on a plane

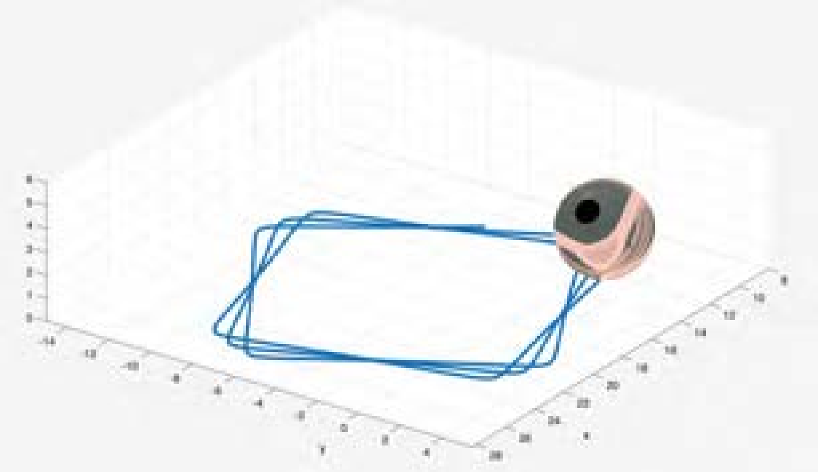
[1] E. J. Routh, The Advanced Part of A Treatise on the Dynamics of a System of Rigid Bodies, 6th ed., Macmillan, London, 1884



Trajectory

Non uniform motion is expected

Simulated motion

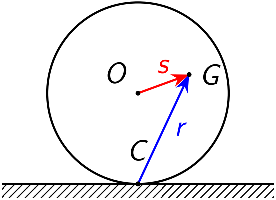


Model and derivation of sensors measurement

Euler equations

$$\mathbf{J} = \mathbf{I}_G + m\left(\|\mathbf{r}\|^2\,\mathbf{E} - \mathbf{r}\,\mathbf{r}^T\right)$$
$$\omega = \mathbf{J}^{-1}\mathbf{M}$$

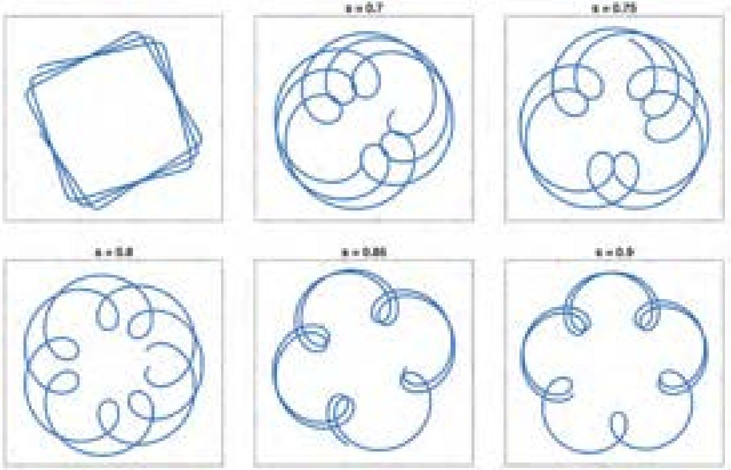
$$\dot{\mathbf{M}} = \mathbf{M} \times \omega + m\left(\omega^T(\mathbf{s} \times \mathbf{r})\right)\omega + m\frac{g}{a}\,\mathbf{r} \times \mathbf{s}$$
$$\dot{\mathbf{r}} = (\mathbf{r} - \mathbf{s}) \times \omega$$
$$\dot{\mathbf{R}} = \mathbf{R}\,[\omega]_\times$$
$$\dot{\mathbf{p}} = -a\,\mathbf{e_z} \times (\mathbf{R}\omega)$$



$s \neq 0$: orientation dependent-torque, non-holonomic constraint (increases apparent inertia),

Spirograph

Spirograph when s is varied



Oscillations in the sensors

Gibbs-Appell equation of motion for the non-holonomic constraints

Routhian

$$R = \frac{1}{2}(\ddot{X}^2 + \ddot{Y}^2 + \ddot{Z}^2) + \frac{1}{2}\sum_i I_i \alpha_i^2$$

Intermediate frame $\mathcal{I}$, (e_1^i, e_2^i, e_3^i) (precession, tilt but no spin), angular velocity is $(\omega_1^i, \omega_2^i, \omega_3^i)^T$

$$\omega_1^i e_1^i + \omega_2^i e_2^i + (\omega_3^i - \dot{\psi})e_3^i = \dot{\theta} e_1^i + \dot{\psi}\sin\theta e_2^i + \dot{\phi}\cos\theta e_3^i$$

Absolute angular acceleration vector is $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with

$$\begin{aligned}
\alpha_1 &= \dot{\omega}_1^i - (\omega_2^i)^2 \cot\theta + \dot{\omega}_2^i \dot{\omega}_3^i \\
\alpha_2 &= \dot{\omega}_2^i + \omega_1^i \omega_2^i \cot\theta - \omega_1^i \omega_3^i \\
\alpha_3 &= \dot{\omega}_3^i
\end{aligned}$$

What does the nutation frequency tells us?

Non-holonomic constraints (rolling without slipping) give

$$\dot{X} = (s\omega_2^i - \omega_3^i)\cos\psi + \omega_1^i \sin\psi(\|\mathbf{s}\|\cos\theta + \sin\theta), \quad \dot{Y} = \ldots$$

Tilt vector in the vicinity of the Euler angle singularity $\theta = 0$

Gibbs-Appell equation gives for the nutation frequency

$$A(s)\,\lambda^2 - K\,\lambda + H(s) = 0, \quad \omega_{\mathrm{nut}} = \frac{\lambda_+(s) - \lambda_-(s)}{2}$$

with $A = I_1 + ms^2, \quad K = (I_3 + ma(a+s))w_3(0), \quad H = mgs$

Want to know the value of s of your Routh sphere?

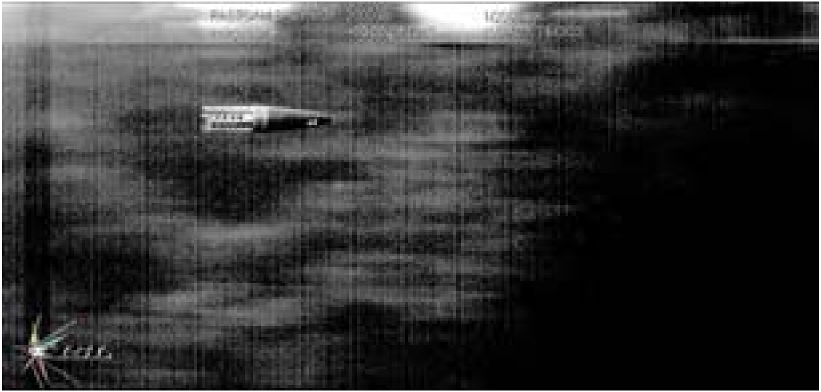
Just spin it fast enough, then watch it roll and measure the frequency of any sensor attached to it.

The real case study: Ballistic wind tunnel (high velocity)

- Gyroscope
- Magnetomoter
- Accelerometer

ExoMars probe

Another version



Estimation strategy

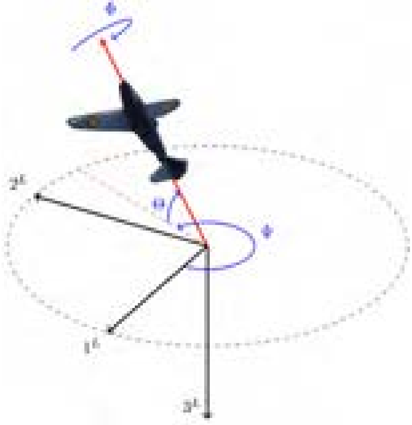
Obtain attitude from magnetometer only

1. If one angle and one direction are known, the complete attitude can be estimated
2. If the velocity is known, the pitch angle can be deduced (via aerodynamics and a gain-scheduled observer).

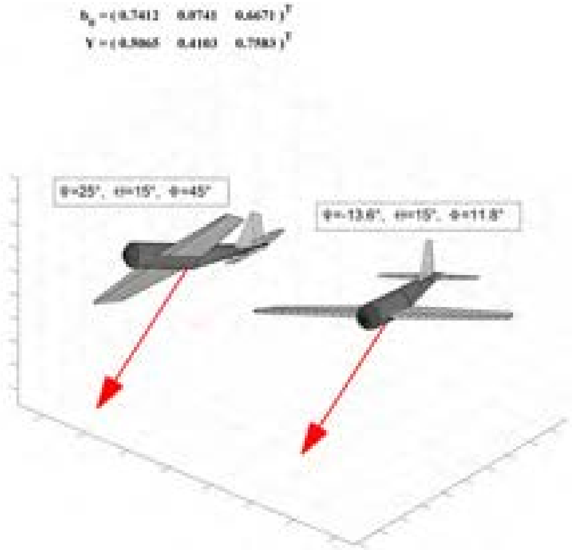
Projectile problem

Only one direction and one angle are measured

Tait-Bryan seq. (ZYX e.g.)
yaw: Ψ
pitch: Θ
roll: Φ



Ambiguity of measurents (b_0) and pitch angles (15^0) are identical



Set of unit quaternions
$\mathbb{Q} = \{q = (s, v \triangleq \mathrm{dir}(q)) \in \mathbb{R} \times \mathbb{R}^3, |q| = 1\}$, with $\otimes$

$$q_1 \otimes q_2 = \begin{pmatrix} s_1 s_2 - v_1^T v_2 \\ s_1 v_2 + s_2 v_1 + v_1 \times v_2 \end{pmatrix}$$

neutral element $(1, 0, 0, 0)$, $q^{-1} = (s, -v)$, $q \otimes q^{-1} = (1, 0, 0, 0)$
Consider

$$R = F(q) := I_3 + 2s\,[v_\times] + 2\,[v_\times]^2 \tag{1}$$

which is "two-to-one" $\mathbb{Q} \rightarrow SO_3$

Measurements and Tait-Bryan angles

Direction measurement

$$Y = \mathrm{dir}\left(q^{-1} \otimes \boldsymbol{p}(b_0) \otimes q\right) \triangleq q^{-1} b_0 q$$

with $\boldsymbol{p}(b_0) \triangleq (0, b_0)$ "pure" quaternion.

$$q \triangleq (q_1, q_2, q_3, q_4) \mapsto \begin{cases} T(q) = q_1 q_3 - q_2 q_4 \\ T_2(q) = q_1 q_4 + q_2 q_3,\ T_3(q) = 1 - 2(q_3^2 + q_4^2) \\ T_4(q) = q_1 q_2 + q_3 q_4,\ T_5(q) = 1 - 2(q_2^2 + q_3^2) \end{cases}$$

$$\Phi(q) = \arctan \frac{2T_4(q)}{T_5(q)}, \quad \Psi(q) = \arctan \frac{2T_2(q)}{T_3(q)}$$

Pitch measurement

$$\Theta(q) = \arcsin 2T(q)$$

Problem summary

Statement

Estimate (in real time) q from $Y = q^{-1} b_0 q$ and $T(q)$.

$$\dot{R} = R\,[w_\times], \quad \dot{q} = \frac{1}{2} q \otimes \boldsymbol{p}(\omega) \tag{2}$$

Solution

$$\begin{cases} \dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \boldsymbol{p}\left(\omega + k_p\sigma\right), \quad \sigma \triangleq Y \times \left(\hat{q}^{-1} b_0\, \hat{q}\right) \\ \dot{\chi} = k_c \dfrac{T(q) - T(\overline{q})}{f\left(\nabla T(\overline{q})^T (\boldsymbol{p}(b_0) \otimes \overline{q})\right)} \\ \overline{q} = \left(\cos\dfrac{\chi}{2} + \sin\dfrac{\chi}{2} b_0\right) \otimes \hat{q} \end{cases} \tag{3}$$

$f : x \mapsto \sqrt{1 + x^2} / \tanh(x)$

$\bar{q}(t)$ converges towards $q(t)$ (conv. to be defined) when $t \to +\infty$

Sketch of proof

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \boldsymbol{p}\left(\omega + k_p\sigma\right)$$

vector error

$$\sigma \triangleq Y \times (\hat{q}^{-1} b_0\, \hat{q}) = (q^{-1} b_0\, q) \times (\hat{q}^{-1} b_0\, \hat{q})$$

Some calculus yields (exponentially, the direction error is zero)

$$\lim_{t \to +\infty} \sigma(t) = 0$$

Extension by integrator on the gradient of pitch error to drive the system through the set $C(q) \triangleq \{\hat{q},\, \sigma(q, \hat{q}) = 0\}$, using an additional (output) rotation

$$\begin{cases} \dot{\chi} = k_c \dfrac{T(q) - T(\overline{q})}{f\left(\nabla T(\overline{q})^T (\boldsymbol{p}(b_0) \otimes \overline{q})\right)} \\ \overline{q} = \left(\cos\dfrac{\chi}{2} + \sin\dfrac{\chi}{2} b_0\right) \otimes \hat{q} \end{cases}$$

Convergence set

$C(q)$ is the intersection of span$\{q, \boldsymbol{p}(b_0) \otimes q\}$ and the unit sphere of $\mathbb{R}^4$, i.e. a circle of $\mathbb{R}^4$. It can be parameterized as

$$C(q) = \{q_\ell \triangleq (\cos\ell)\, q + (\sin\ell)\,(\boldsymbol{p}(b_0) \otimes q)\,,\ \ell \in [0, 2\pi]\} \qquad (4)$$

Intersection of $C(q)$ with $T(q_\ell) = T(q)$ gives (with $b_0 \triangleq [a\, b\, c]^T$)

$$\begin{aligned} T(q_\ell) = T(q) = &(\cos^2\ell)\, T(q) + (\cos\ell \sin\ell)(bT_3(q) - 2aT_2(q)) \\ &+ (\sin^2\ell)(-acT_3(q) - 2bcT_2(q)) \\ &+ (\sin^2\ell)(c^2 - a^2 - b^2)\, T(q) \end{aligned}$$

which has 2 isolated solutions $\ell \in [0, \pi[$: $\ell = 0$ and

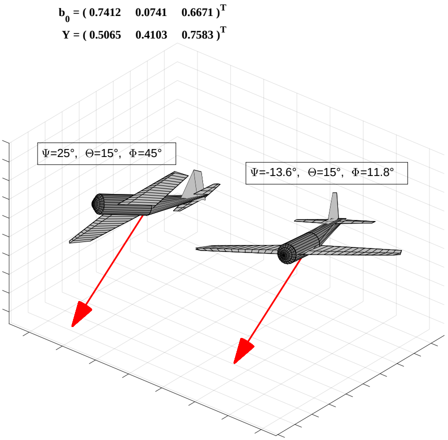
$$\ell_\# = \frac{\pi}{2} + \arctan\left(\frac{bT_3(q) - 2aT_2(q)}{2(a^2 + b^2)\, T(q) + acT_3(q) + 2bcT_2(q)}\right)$$

$q \neq q_\#$

(and $\ell = \pi$ and $\ell = \ell_\# + \pi$, corresponding to $-q$ and $-q_\#$ (yielding identical rotations in SO_3).

Proposition

There exist *4 elements in $C(q)$* *possessing the same pitch angle as q. These 4 are q, $-q$, $q_{\#}$ and $-q_{\#}$ yielding two rotations.*



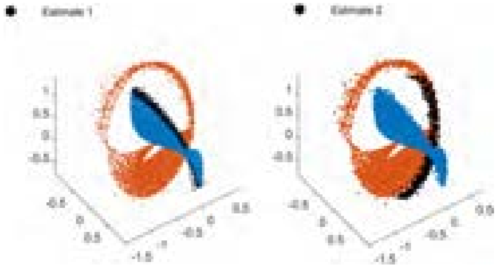
Asymptotic convergence

Theorem (cv. towards one of the two possible attitudes)

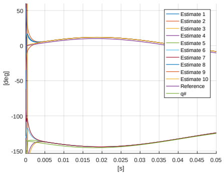
Let $\hat{q}(0)$ *and* $\chi(0)$ *s.t.* $\overline{q}(0) \notin E_0$. *There exists* $\eta_2 > 0$ *s.t., for any* $\varepsilon > 0$ *small enough, there exist* $\eta_1 > 0$ *and* $k_c* > 0$ *s.t., if* $\|\sigma(0)\| < \eta_1$, $|T(q(0)) - T(\overline{q}(0))| < \eta_2$ *and* $k_c > k_c*$, *then there exists* $K, \lambda > 0$ *s.t. for any* $t \geq 0$, *one has*
$\min\left(|\delta(q,\overline{q})|, |\delta(-q,\overline{q})|, |\delta(q_{\#},\overline{q})|, |\delta(-q_{\#},\overline{q})|\right) < \varepsilon + K\exp^{-\lambda t}$.

$\delta(.,.)$: angle of relative quaternion

Practical behavior



Stereo. view of q, $q_{\#}$
and estimate $\overline{q}$ for various initial conditions.



Yaw angle

Straight-forward clarification

$\sigma(t)$ goes to zero at t$\rightarrow \infty$. Then $\overline{q}$ remains on side of the separating hyperplane. The projectile does not fly backwards.

Particular case

Pitch angle and Earth's magnetic field enable differentiation. (except if the flight happens in a pure "North-South" plane).

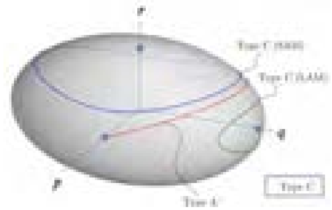
② Estimating the velocity: equation of "exterior ballistics"

Euler-Poinsot equations (free rotation) with torques (Magnus $\mathcal{M}$, overturning $\mathcal{O}$, drag $\mathcal{D}$)

$$\begin{cases} I_l \dot{p} = -\mathcal{D}_p \, v \, p \\ I_t \dot{q} = (I_l - I_t) \, p \, r + \mathcal{M} \, v \, \beta + \mathcal{O} \, v^2 \, \alpha - \mathcal{D} \, v \, q \\ I_t \dot{r} = (I_l - I_t) \, p \, q + \mathcal{M} \, v \, \alpha + \mathcal{O} \, v^2 \, \beta - \mathcal{D} \, v \, r \end{cases} \tag{5}$$

Type C reduction (intermediate frame),
parametric oscillator

$$\ddot{\xi} + v(H - i\,P)\dot{\xi} - v^2(M + i\,PT)\xi = -i\,PG$$



H, M, P, T, G: aero.

$$\xi(t) = A_n e^{iw_n t} + A_p e^{iw_p t} + A_0$$

Roll, Precession (rot. about roll axis),
Nutation (tilt of roll axis)

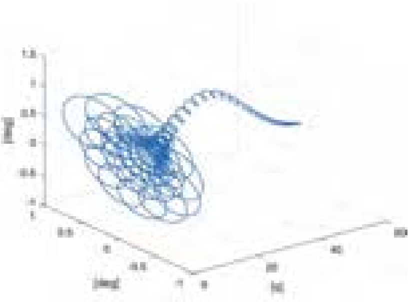
Angle of attack (aerodynamics)

Body attached epicyclic motion

$$\alpha(t) + i\,\beta(t) = -i\,(A_n e^{i(p-w_n)t} + A_p e^{i(p-w_p)t} + A_0 e^{ipt})$$

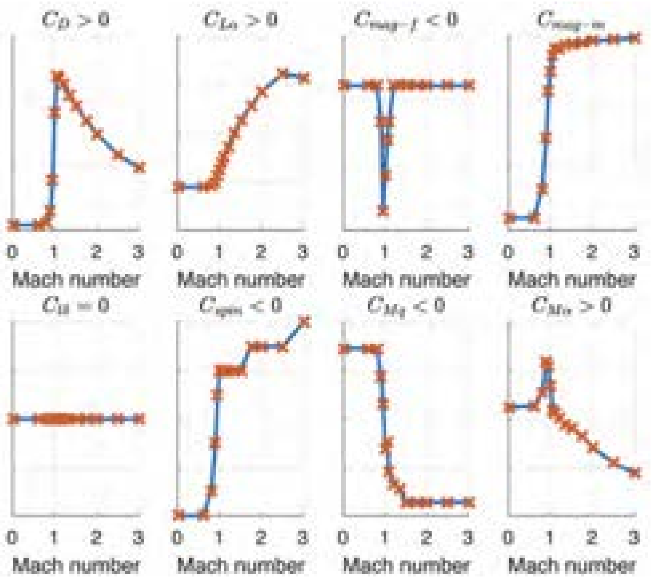
$$w_n = p\frac{I_l}{2I_t} + \frac{v}{2}(P_1^2 + P_2^2)^{\frac{1}{4}} \cos(\frac{1}{2}\arctan(\frac{P_2}{P_1}))$$

$$w_p = p\frac{I_l}{2I_t} - \frac{v}{2}(P_1^2 + P_2^2)^{\frac{1}{4}} \cos(\frac{1}{2}\arctan(\frac{P_2}{P_1}))$$

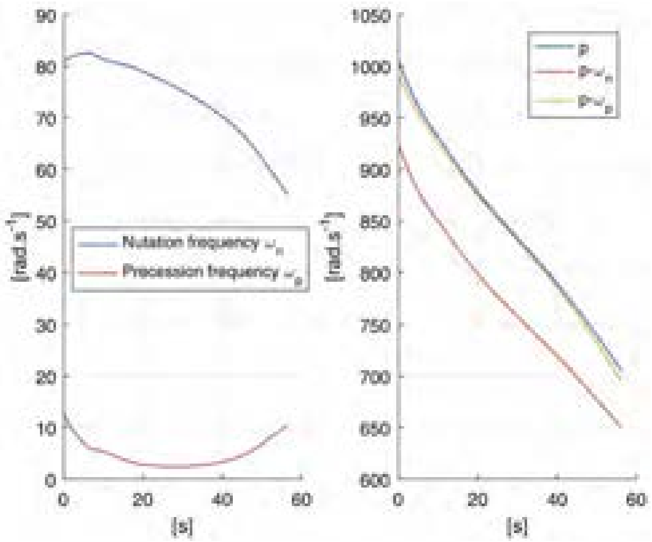


Analytic formula from aerodynamic forces and torques reference tables (drag, Magnus, lift).

Aerodynamic reference tables

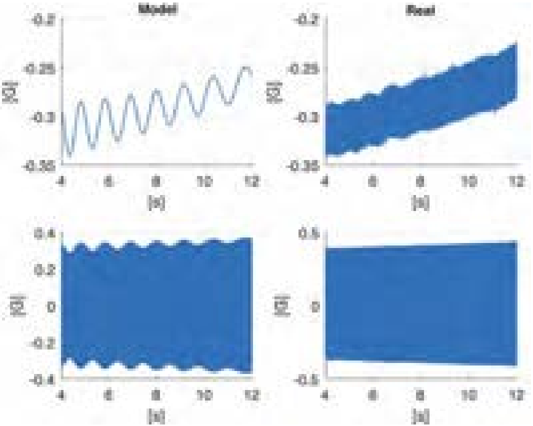


Theoretical frequencies



Magnetometer measurements

- Eddy currents
- Centrifugal effects
- Crosstalks
- Mis-alignments



"Vector space" frequency detection

$$s(t) = \sum_{k=1}^{p} \alpha_k \exp(i(w_k t + \phi_k)) + n(t)$$

$$y(t) = [s(t), ..., s(t+m-1)]^T$$

$$C = E(y(t)\, y^*(t)) \in \mathbb{C}^{m\times m} = APA^* + \sigma^2 I_m$$

$$P = \mathrm{diag}(\alpha_1^2, ..., \alpha_p^2), \quad A = (a(w_1), ..., a(w_p)) \in \mathbb{C}^{m\times p}$$

$$a(w) = (1, \exp(iw), ..., \exp(iw(m-1))^T$$

$\mathrm{eig}(C) : \lambda_1 \geq ... \geq \lambda_p > \lambda_{p+1} = ... = \lambda_m = \sigma^2 : G = [g_1, ..., g_{m-p}]$

Discriminating algebraic function

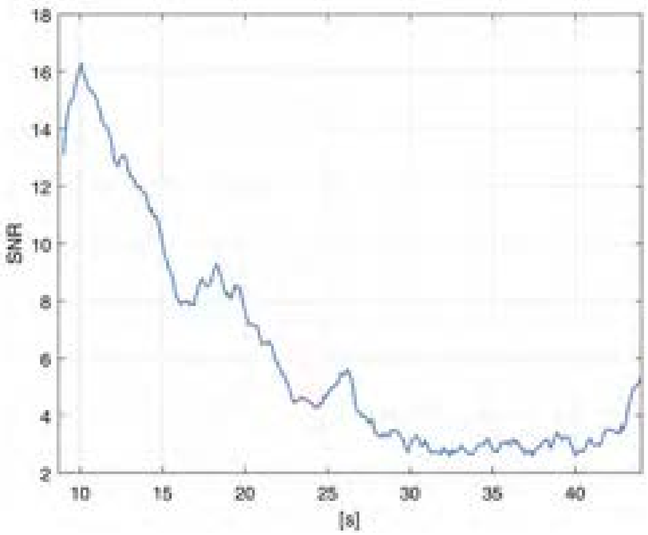
$$h_{\mathrm{MU}}(w) = \frac{1}{\sum_{i=1}^{m-p} \left\| g_i^* a(w) \right\|^2} = \frac{1}{a^2(w) G G^* a(w)}$$

Detection of p peaks

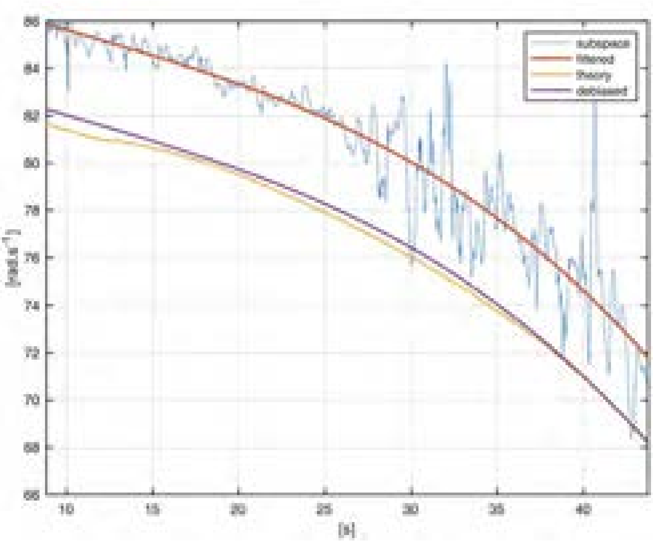
$$a^*(w_i) G G^* a(w_i) = 0, \quad i = 1, ..., p$$

Recursive, embeddable implementation

Application: direct fire 40 s, 300 m/s, 900 rad/s

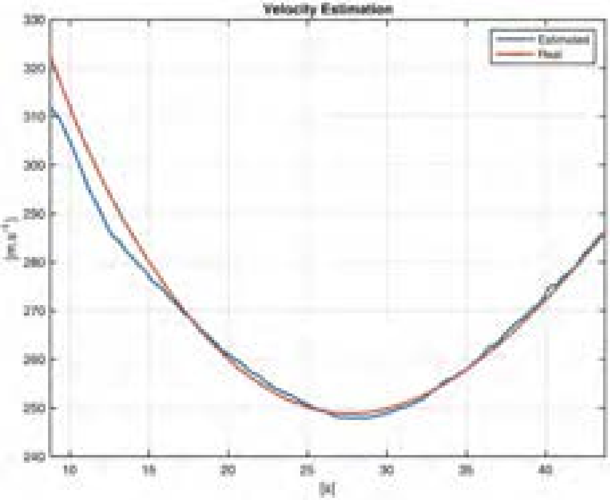


Dectected frequencies



after roll-deconvolution

Velocity estimation



First velocity estimation without radar, GPS, or vision

Thank you for your attention